ON GENERALIZATIONS OF ANABELIAN GROUP-THEORETIC PROPERTIES

ARATA MINAMIDE, KOICHIRO SAWADA, AND SHOTA TSUJIMURA

Abstract. In the present paper, we discuss certain generalizations on two anabelian group-theoretic properties — strong internal indecomposability [i.e., the property that for every open subgroup H, the centralizer in H of every nontrivial normal closed subgroup N of H is trivial] and elasticity [i.e., the property that every nontrivial topologically finitely generated normal closed subgroup of an open subgroup is open]. More concretely, by replacing the normality conditions appearing in characterizations of strong internal indecomposability and elasticity by the subnormality conditions, we introduce the notions of strong sn-internal indecomposability and sn-elasticity and prove that various profinite groups appearing in anabelian geometry satisfy these properties.

Contents

Introduction

For any connected Noetherian scheme S, we shall write $\pi_1(S)$ for the étale fundamental group of S, relative to a suitable choice of basepoint. For any field K, we shall write G_K for the absolute Galois group of K, relative to a suitable choice of separable closure. Let p be a prime number.

Roughly speaking, one of the main motivations of anabelian geometry is

> *to specify the class of anabelian varieties, i.e., algebraic varieties X which may be "reconstructed" from $\pi_1(X)$.*

For instance, if X is a hyperbolic curve over a p-adic local field [i.e., a finite extension field of the field of p-adic numbers] or a number field [i.e., a finite extension field of the field of rational numbers], then Nakamura, Tamagawa, and Mochizuki proved that X may be "reconstructed" from $\pi_1(X)$ [cf. [24], Theorem 1.1; [32], Theorem

2020 *Mathematics Subject Classification.* Primary 14H30; Secondary 12E30, 20E15.

Key words and phrases. subnormal subgroup; profinite group; internal indecomposability; elasticity; absolute Galois group; surface group.

0.4; [17], Theorem A; [21], Introduction]. However, in the case of higher-dimensional algebraic varieties, it seems far-reaching to solve this problem.

With regard to this problem, since we need to "reconstruct" X from *only the profinite group* $\pi_1(X)$, it is natural to expect that $\pi_1(X)$ has nice/special group-theoretic properties. Thus, in this context, it would be natural to study *anabelian group-theoretic properties*, i.e.,

(I) distinctive group-theoretic properties which should be satisfied by the *geometric fundamental group* of every anabelian variety X [i.e., the étale fundamental group of the algebraic variety obtained by base-changing X to an algebraic closure of the base field of X], or

(II) distinctive group-theoretic properties which need not to be satisfied by the geometric fundamental group of every anabelian variety X, but *play important roles* in anabelian geometry.

One (and perhaps the only one) widely-accepted example of anabelian group-theoretic properties of type (I) is

- *slimness* — i.e., the property that the center of every open subgroup is trivial.

Note that [as is well-known] the geometric fundamental group of any hyperbolic curve over a number field or a p-adic local field satisfies slimness. On the other hand, examples of anabelian group-theoretic properties of type (II) are

- *strong internal indecomposability* — i.e., the property that for every open subgroup H, the centralizer in H of every nontrivial normal closed subgroup N of H is trivial, and
- *elasticity* — i.e., the property that every nontrivial topologically finitely generated normal closed subgroup of an open subgroup is open.

Indeed, the strong internal indecomposabilities of certain profinite groups — which, in fact, will be proved later [cf. Theorem C, (i), below] — may be regarded as generalizations of famous *injectivity results* [cf. [2], Theorem 1; [6], Theorem C, (ii)] in anabelian geometry by the following [easily verified] fact:

Fact. *Let G be a strongly internally indecomposable profinite group; $N \subseteq G$ a nontrivial normal closed subgroup. Then the natural outer representation $G/N \to \mathrm{Out}(N)$ is injective. [Here, we use the notation "$\mathrm{Out}((-))$" to denote the group of outer continuous automorphisms of $(-)$.]*

Moreover, since various highly nontrivial outer representations [cf., e.g., the outer representations appearing in [2], Theorem 1; [6], Theorem C, (ii)] play central roles and pose various important questions in anabelian geometry, it appears to the authors that this property could be used to pose new interesting questions in anabelian geometry. On the other hand, the elasticities of "pro-$\mathcal{C}$ surface groups" [cf. Definition 3.12; [22], Theorem 1.5] play essential roles in the study of *higher-dimensional anabelian varieties* [cf., e.g., [4], Theorem B; [5], Theorem A; [22], Corollary 6.3; [29], Theorem 1.2; [30], Theorem C].

Note that the geometric fundamental group of any affine hyperbolic curve over a number field or a p-adic local field satisfies strong internal indecomposability and elasticity [cf. Theorem 1, below]. Also, note that [as is easily verified] the geometric fundamental group of the fiber product [over the base field] of two copies of any hyperbolic curve over a number field or a p-adic local field — which may be regarded as an example of *two-dimensional anabelian varieties* — does not satisfy strong internal indecomposability and elasticity.

In the present paper, we focus on anabelian group-theoretic properties of type (II), especially, strong internal indecomposability and elasticity. With regard to these two properties, for instance, the following results are known:

Theorem 1 ([26], Theorem 8.6.6; [26], Proposition 8.7.8)**.** *Let $\mathcal{C}$ be a nontrivial full-formation [i.e., a family of finite groups which contains the trivial group, and which is closed under taking quotients, subgroups, and extensions]; F a free pro-$\mathcal{C}$ group of [possibly infinite] rank ≥ 2. Then F is strongly internally indecomposable and elastic.*

Theorem 2 ([12], Theorem 2.1; [13], Theorem C; [14], Theorem A)**.** *Suppose that the field K satisfies one of the following conditions:*

- *K is a Henselian discrete valuation field of residue characteristic p;*
- *K is a Hilbertian field [i.e., a field for which Hilbert's irreducibility theorem holds].*

Then G_K is strongly internally indecomposable and elastic.

Considering these two results, it is natural to pose the following question:

Question 1: We continue to use the notation of Theorems 1, 2.
- (i) Can one find new examples of profinite groups Γ appearing in anabelian geometry which satisfy strong internal indecomposability (respectively, elasticity)?
- (ii) Do F, G_K, and Γ satisfy stronger properties than strong internal indecomposability (respectively, elasticity)?

Let us give an explicit version of Question 1, (ii). Let G be a profinite group;

$$\mathcal{P}(G)$$

a group-theoretic condition concerning closed subgroups of G [i.e., such as "normal in G" or "not procyclic"]. In this Introduction, we shall say that G is

- *strongly $\mathcal{P}$-internally indecomposable* if for every open subgroup H, the centralizer in H of every nontrivial $\mathcal{P}(H)$-closed subgroup N of H is trivial;
- *$\mathcal{P}$-elastic* if every nontrivial topologically finitely generated $\mathcal{P}(G)$-closed subgroup of an open subgroup is open.

[Here, observe that, in the case where

$$\mathcal{P}(G) \ = \ \{\text{equal to } G\},$$

the notion of strong $\mathcal{P}$-internal indecomposability coincides with the notion of slimness. Moreover, note that, in the case where

$$\mathcal{P}(G) \ = \ \{\text{normal in } G\},$$

the notion of strong $\mathcal{P}$-internal indecomposability (respectively, $\mathcal{P}$-elasticity) coincides with the notion of strong internal indecomposability (respectively, elasticity).] Then one of the explicit versions of Question 1, (ii), is as follows:

Question 2: In the notation of Question 1, can one find a *weakest* condition $\mathcal{P}(G)$ such that F, G_K, and Γ satisfy strong $\mathcal{P}$-internal indecomposablity (respectively, $\mathcal{P}$-elasticity)?

In the present paper, as a first step toward answering these questions, we treat the case where

$$\mathcal{P}(G) \ = \ \{\textit{subnormal} \text{ in } G\}.$$

[We shall say that a closed subgroup $H \subseteq G$ is subnormal in G if there exist a nonnegative integer m and [not necessarily distinct] closed subgroups $H_0 = G$, H_1, $\dots$, H_{m-1}, $H_m = H$ of G such that H_i is normal in H_{i-1} for each $i \in \{1, \dots, m\}$.]

Our main results are the following:

Theorem A (Theorems 3.3, 3.7, 3.13)**.** *Let $\mathcal{C}$ be a nontrivial full-formation. Suppose that a profinite group F satisfying one of the following conditions:*

- *F is an almost pro-$\mathcal{C}$-maximal quotient [cf. Definition 2.7, (iii)] of a free profinite group of [possibly infinite] rank ≥ 2;*
- *F is an almost pro-$\mathcal{C}$-maximal quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic 0;*
- *F is a pro-p Demuškin group of rank ≥ 3 [cf. Definition 3.5].*

Then F is strongly sn-internally indecomposable [cf. Definition 1.8, (iii)] and sn-elastic [cf. Definition 2.1, (ii)].

Theorem B (Theorems 3.8, 3.10)**.** *Let $\mathcal{C}$ be a full-formation such that $\mathbb{Z}/p\mathbb{Z}$ belongs to $\mathcal{C}$. Suppose that the field K satisfies one of the following conditions:*

- *K is a Henselian discrete valuation field of residue characteristic p;*
- *K is a Hilbertian field.*

Then any almost pro-$\mathcal{C}$-maximal quotient of G_K is strongly sn-internally indecomposable and sn-elastic.

Theorem C (Theorems 3.16, 3.19; Theorem 3.20, (i), (iii))**.** *Let n be a positive integer; K a field; X a hyperbolic curve over K. Write $\overline{K}$ for the algebraic closure [determined up to isomorphisms] of K; X_n for the n-th configuration space associated to X [cf. Definition 3.15, (i)]. Then the following hold:*

(i) *Suppose that the field K satisfies one of the following conditions:*
  - *K is an algebraically closed field of characteristic 0;*
  - *K is a number field;*
  - *K is a p-adic local field.*

  Then $\pi_1(X_n)$ is strongly sn-internally indecomposable.

(ii) *Let Σ be a set of prime numbers which is either equal to the set of all prime numbers or equal to the set of all prime numbers $\neq p$. Suppose that the field K satisfies one of the following conditions:*
  - *K is an algebraically closed field of characteristic p;*
  - *K is a finite field of characteristic p.*

  Then the geometrically pro-Σ fundamental group

$$\pi_1(X)^{[\Sigma]} \ \stackrel{\text{def}}{=} \ \pi_1(X)/\ker(\pi_1(X \times_K \overline{K}) \twoheadrightarrow \pi_1(X \times_K \overline{K})^{\Sigma})$$

  — where we use the notation "$(-)^{\Sigma}$" to denote the maximal pro-Σ quotient of $(-)$ — of X is strongly sn-internally indecomposable.

Theorem C, together with Theorem A; the "non-hyperbolic" cases of Theorem 3.19 and Theorem 3.20, (iii), may be regarded as partial generalizations of [12], Theorem A; [12], Corollary D.

Here, note that the n-th configuration spaces associated to hyperbolic curves over number fields or p-adic local fields may be regarded as examples of *n-dimensional* *anabelian varieties*. Also, we remark that, in the notation of Theorem C, if $n \geq 1$ is an arbitrary integer and K is the field of complex numbers (respectively, $n = 1$ and K is a p-adic local field), then a similar result to the result stated in Theorem C, (i), holds for the *topological fundamental group* of the complex analytic space associated to X_n [cf. Corollary 3.17] (respectively, *tempered fundamental group* of the Berkovich space associated to X [cf. Theorem 3.20, (ii)]). [On the other hand, in light of further development, it may be worth mentioning that since the properties themselves we consider in the present paper are purely group-theoretic, there is no reason to restrict our attention to topological groups that appear in anabelian geometry.]

The strong $\mathcal{P}$-internal indecomposablity case of Question 2 is also interesting from the point of view of the following natural question [in anabelian geometry], which concerns the "*relative version of the Grothendieck Conjecture for hyperbolic curves over a field K of characteristic* 0" (RGC_K) — i.e., a conjecture to the effect that for any hyperbolic curves C_1, C_2 over K, the natural map from the set of K-isomorphisms between C_1 and C_2 to the set of G_K-isomorphisms between $\pi_1(C_1)$ and $\pi_1(C_2)$, considered up to composition with an inner automorphism arising from the geometric fundamental group of C_2, is bijective:

> Question 3: Let K be a number field or a p-adic local field; $K \subseteq L$ an algebraic field extension. [In particular, (RGC_K) holds — cf. [17], Theorem A.] Then can one find a *weakest* condition on L such that (RGC_L) holds?

Indeed, suppose that (RGC_L) holds. Then, by considering the case where "$C_1 = C_2 =$ the projective line [over K or L] minus 0, 1, ∞", one verifies immediately that the centralizer of G_L in G_K is *trivial* [cf. [6], Theorem C, (ii); [32], Lemma 7.1; [32], Remark 7.3, (i)].

Finally, the authors hope to be able to propose

> a *new candidate* for anabelian group-theoretic property of type (I)

in our subsequent paper.

Note: The sn-internal indecomposability (respectively, sn-elasticity) portion of Theorem B is applied in the proof of [16], Theorem A, (i) (respectively, [23], Theorem 3.11, (i)), to obtain a certain *anabelian result*.

Notations and conventions

Numbers: The notation $\mathbb{Z}$ will be used to denote the ring of integers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of the underlying additive group of $\mathbb{Z}$. The notation $\mathbb{C}$ will be used to denote the field of complex numbers.

If p is a prime number, then the notation $\mathbb{Z}_p$ will be used to denote the ring of p-adic integers; the notation $\mathbb{F}_p$ will be used to denote the finite field of cardinality p; the notation $\mathbb{C}_p$ will be used to denote the p-adic completion of an algebraic closure of the field of fractions of $\mathbb{Z}_p$.

We shall refer to a finite extension field of the field of fractions of $\mathbb{Z}$ as a *number field*. We shall refer to a finite extension field of the field of fractions of $\mathbb{Z}_p$ as a *p-adic local field*.

Fields: Let F be a field; F^{sep} a separable closure of F. Then we shall write $G_F \overset{\text{def}}{=} \text{Gal}(F^{\text{sep}}/F)$. If F is perfect, then we shall also write $\overline{F} \overset{\text{def}}{=} F^{\text{sep}}$.

Schemes: Let K be a field; $L \supseteq K$ a field extension; X a scheme over K. Then we shall write $X_L \overset{\text{def}}{=} X \times_K L$.

Groups: Let G be a group; $H \subseteq G$ a subgroup. Then we shall write $[G : H]$ for the index of H in G; $Z_G(H)$ for the *centralizer* of H in G, i.e., the subgroup $\{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$; $Z(G) \overset{\text{def}}{=} Z_G(G)$; $N_G(H)$ for the *normalizer* of H in G, i.e., the subgroup $\{g \in G \mid gHg^{-1} = H\}$. We shall say that G is *center-free* if $Z(G) = \{1\}$. We shall write $\text{Out}(G)$ for the group of outer automorphisms of G, i.e., the quotient of the group of automorphisms of G by the normal subgroup of inner automorphisms of G. [If G is profinite, then we consider automorphisms in the category of profinite groups.]

For subgroups $H_1, H_2 \subseteq G$ of G, we shall write $[H_1, H_2]$ for the subgroup of G generated by $\{[a, b] \mid a \in H_1, b \in H_2\} \subseteq G$, where $[a, b] \overset{\text{def}}{=} aba^{-1}b^{-1}$.

If G is a topological group, then for a subset $S \subseteq G$ of G, we shall write $\overline{S}$ for the closure of S in G.

Suppose that G is a profinite group. Then we shall say that G is *slim* if $Z_G(U) = \{1\}$ for every open subgroup U of G, or, equivalently, every open subgroup of G is center-free [cf. [21], Notations and Conventions]. If G is a topologically finitely generated, then we shall write $r(G)$ for the minimum number of topological generators of G. If G is not topologically finitely generated, then $r(G) \overset{\text{def}}{=} \infty$. We shall refer to $r(G)$ as the *rank* of G.

Fundamental groups: Let S be a connected locally Noetherian scheme. Then we shall write $\pi_1(S)$ for the étale fundamental group of S, relative to a suitable choice of basepoint. [Note that, for any field F, $\pi_1(\text{Spec}\, F) \cong G_F$.] If X is an algebraic variety [i.e., a separated, of finite type, and geometrically connected scheme] over $\mathbb{C}$, then we shall write $\pi_1^{\text{top}}(X)$ for the topological fundamental group of the complex analytic space associated to X, relative to a suitable choice of [$\mathbb{C}$-rational] basepoint. If K is a complete subfield of $\mathbb{C}_p$ and X is a smooth variety over K, then we shall write $\pi_1^{\text{temp}}(X)$ for the tempered fundamental group of the Berkovich space associated to X, relative to a suitable choice of basepoint [cf. [1]].

1. Internal indecomposability of subnormal subgroups

In the present section, we discuss generalities on *internal indecomposability* [cf. Definition 1.5] of *subnormal subgroups* [cf. Definition 1.1] of [not necessarily profinite] groups. In particular, we obtain generalizations of almost all of the results of [14], §1 [cf. Propositions 1.7, 1.12, 1.14, 1.16, 1.17; Lemma 1.15].

Definition 1.1. Let G be a group; $H \subseteq G$ a subgroup; n a nonnegative integer. Then we shall say that H is *n-subnormal* in G if there exist [not necessarily distinct] subgroups $H_0 = G, H_1, \ldots, H_{n-1}, H_n = H$ of G such that H_i is normal in H_{i-1} for each $i \in \{1, \ldots, n\}$. We shall say that H is *subnormal* in G if there exists a nonnegative integer m such that H is m-subnormal in G.

Remark 1.1.1. If G is a topological group and $H, H' \subseteq G$ are subgroups such that H is normal in H', then $\overline{H}$ is normal in $\overline{H'}$. In particular, if H is an n-subnormal closed subgroup of a topological group G, then we may choose H_i in Definition 1.1 to be closed.

Remark 1.1.2. If $H \subseteq G$ is a normal subgroup, then it is clear that $Z_G(H) \subseteq G$ is normal in G. However, if $H \subseteq G$ is a subnormal subgroup of G, then $Z_G(H)$ is not necessarily subnormal in G. For instance, the symmetric group $\mathfrak{S}_4$ on 4 letters has subgroups

$$V \stackrel{\text{def}}{=} \{\text{id}, (12)(34), (13)(24), (14)(23)\} \subseteq \mathfrak{S}_4, \ H \stackrel{\text{def}}{=} \{\text{id}, (12)(34)\} \subseteq V.$$

[V is known as the *Klein four-group*.] Since H is normal in V and V is normal in $\mathfrak{S}_4$, H is a subnormal subgroup of $\mathfrak{S}_4$. On the other hand, one may easily confirm that

$$Z_{\mathfrak{S}_4}(H) = N_{\mathfrak{S}_4}(H) = \{\text{id}, (12)(34), (13)(24), (14)(23), (12), (34), (1324), (1423)\}$$

is not a subnormal subgroup of $\mathfrak{S}_4$.

Proposition 1.2 ([3], Lemma 1.2.5, (b))**.** *Let G be a profinite group; $H \subseteq G$ a closed subgroup; $V \subseteq H$ an open subgroup of H. Then there exists an open subgroup $U \subseteq G$ such that $U \cap H = V$.*

Corollary 1.3. *Let G be a profinite group; $H \subseteq G$ a closed subgroup; n a nonnegative integer. Consider the following conditions:*

1. *H is an open subgroup of a subnormal closed subgroup of G.*
2. *There exist nonnegative integer m and closed subgroups $H_0 = G, H_1, \ldots, H_{m-1}, H_m = H$ of G such that for each $i \in \{1, \ldots, m\}$, H_i is a subgroup of H_{i-1} which is open or normal.*
3. *H is a subnormal closed subgroup of an open subgroup of G.*

Then we have an implication (1) $\Rightarrow$ (2) and an equivalence (2) $\Leftrightarrow$ (3). Moreover, if H is an open subgroup of an n-subnormal closed subgroup of G, then H is an n-subnormal closed subgroup of an open subgroup of G.

Proof. This follows immediately from Proposition 1.2. $\square$

Proposition 1.4. *Let G be a group; n a positive integer; $H_0 = G, H_1, \ldots, H_{n-1}, H_n$ subgroups of G such that H_i is normal in H_{i-1} for each $i \in \{1, \ldots, n\}$. Then the equality $Z_G(H_n) = \{1\}$ holds if and only if for each $i \in \{1, \ldots, n\}$, it holds that $Z_{H_{i-1}}(H_i) = \{1\}$.*

Proof. Necessity is immediate. Sufficiency is [a special case of] [27], 13.5.3. $\square$

Definition 1.5 (cf. [14], Definition 1.1, (iii), (iv), (v))**.** Let G be a group.

(i) Let $H \subseteq G$ be a subgroup. We shall say that H is *normally decomposable* in G if there exist nontrivial normal subgroups $H_1, H_2 \subseteq G$ of G such that $H = H_1 \times H_2$. We shall say that H is *normally indecomposable* in G if H is not normally decomposable in G. We shall say that G is *decomposable* (respectively, *indecomposable*) if G is normally decomposable (respectively, normally indecomposable) in G.

(ii) We shall say that G is *internally indecomposable* if every normal subgroup of G is center-free and normally indecomposable in G. If G is a profinite group, then we shall say that G is *strongly internally indecomposable* if every open subgroup of G is internally indecomposable.

Remark 1.5.1. [14], Definition 1.1 deals with the case where G is profinite. Since the various subgroups of G are assumed to be closed in [14], Definition 1.1, the definitions here are a priori different from the definitions in [14], Definition 1.1. Note that a profinite group is internally indecomposable (respectively, strongly internally indecomposable) in the sense of Definition 1.5, (ii), if and only if the profinite group is internally indecomposable (respectively, strongly internally indecomposable) in the sense of [14], Definition 1.1, (v) [cf. Proposition 1.7 below; [14], Proposition 1.2].

Lemma 1.6. *Let G be a Hausdorff topological group; $S \subseteq G$ a nonempty subset. Then the following hold:*

(i) $Z_G(S) \subseteq G$ *is closed.*
(ii) $Z_G(S) = Z_G(\overline{S})$.

Proof. First, we verify assertion (i). For any $h \in G$, $Z_G(\{h\})$ is the inverse image of the closed subset $\{1\} \subseteq G$ via the continuous map $G \to G$ determined by $g \mapsto ghg^{-1}h^{-1}$. Thus, $Z_G(\{h\})$, hence also $Z_G(S) = \bigcap_{h \in S} Z_G(\{h\})$, is closed. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since $S \subseteq \overline{S}$, the inclusion $Z_G(S) \supseteq Z_G(\overline{S})$ is clear. We verify the inclusion $Z_G(S) \subseteq Z_G(\overline{S})$. Let us observe that S is contained in $Z_G(Z_G(S))$. Now it follows from assertion (i) that $\overline{S} \subseteq Z_G(Z_G(S))$. Thus, it holds that $Z_G(S) \subseteq Z_G(\overline{S})$. This completes the proof of assertion (ii), hence also of Lemma 1.6. □

Proposition 1.7. *Let G be a group; n a positive integer. Then the following conditions are equivalent:*

(1) *Every $(n-1)$-subnormal subgroup of G is internally indecomposable.*
(2) $Z_G(H) = \{1\}$ *for every nontrivial n-subnormal subgroup $H \subseteq G$.*

Moreover, if G is a Hausdorff topological group, then the above conditions and the following conditions are all equivalent:

(3) *Every $(n-1)$-subnormal closed subgroup of G is internally indecomposable.*
(4) $Z_G(H) = \{1\}$ *for every nontrivial n-subnormal closed subgroup $H \subseteq G$.*

Proof. First, we verify the implication (1) $\Rightarrow$ (2). Suppose that condition (1) is satisfied. Let $H \subseteq G$ be a nontrivial n-subnormal subgroup. Then there exist subgroups $H_0 = G, H_1, \ldots, H_{n-1}, H_n = H$ of G such that H_i is normal in H_{i-1} for each $i \in \{1, \ldots, n\}$. Now for each $i \in \{1, \ldots, n\}$, since H_{i-1} is internally indecomposable, it follows from [the proof of] [14], Proposition 1.2, that $Z_{H_{i-1}}(H_i) = \{1\}$. [Note that the proof of [14], Proposition 1.2, is also valid in the case of general groups.] Now it follows from Proposition 1.4 that $Z_G(H) = \{1\}$. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. Let $H \subseteq G$ be an $(n-1)$-subnormal subgroup. Then, for every nontrivial normal subgroup of $N \subseteq H$ of H, it holds that $Z_H(N) \subseteq Z_G(N) = \{1\}$. Thus, it follows from [the proof of] [14], Proposition 1.2, that H is internally indecomposable. This completes the proof of the implication (2) $\Rightarrow$ (1).

Finally, we verify the equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) when G is a Hausdorff topological group. The equivalence (3) $\Leftrightarrow$ (4) follows from an argument similar to the above argument. Moreover, the equivalence (2) $\Leftrightarrow$ (4) follows from Remark 1.1.1; Lemma 1.6. This completes the proof of Proposition 1.7. □

Definition 1.8. Let G be a group; n a positive integer.

(i) We shall say that G is *n-sn-internally indecomposable* if G satisfies the equivalent conditions (1), (2) of Proposition 1.7. [Note that a 1-sn-internally indecomposable group is nothing but an internally indecomposable group.]

(ii) If G is a profinite group, then we shall say that G is *strongly n-sn-internally indecomposable* if every open subgroup of G is n-sn-internally indecomposable.

(iii) We shall say that G is *sn-internally indecomposable* (respectively, *strongly sn-internally indecomposable*) if G is m-sn-internally indecomposable (respectively, strongly m-sn-internally indecomposable) for any positive integer m.

Remark 1.8.1.

(i) Let m, n be integers such that $n \geq m \geq 0$. If a group (respectively, a profinite group) G is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable), then any m-subnormal subgroup (respectively, m-subnormal closed subgroup) of G is $(n-m)$-sn-internally indecomposable (respectively, strongly $(n-m)$-sn-internally indecomposable) [cf. Corollary 1.3].

(ii) A group (respectively, a profinite group) is sn-internally indecomposable if and only if every subnormal subgroup (respectively, subnormal closed subgroup) is center-free and indecomposable.

(iii) The following Proposition 1.9 shows that 2-sn-internal indecomposability is a strictly stronger condition than internal indecomposability. Note that the group $G^n \rtimes \mathfrak{S}_n$ appearing in Proposition 1.9 is often referred to as the wreath product of G by $\mathfrak{S}_n$.

Proposition 1.9. *Let n be an integer such that $n \geq 2$; G a nontrivial group. Write $\mathfrak{S}_n$ for the symmetric group on n letters. We define an action of $\mathfrak{S}_n$ on G^n by*

$$\sigma(g_1, \ldots, g_n) = (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}) \ (\sigma \in \mathfrak{S}_n, (g_1, \ldots, g_n) \in G^n).$$

Then the following hold:

(i) *Let $N \subseteq G^n \rtimes \mathfrak{S}_n$ be a normal subgroup of $G^n \rtimes \mathfrak{S}_n$. Suppose that $N \cap G^n = \{1\}$. Then it holds that $N = \{1\}$.*

(ii) *Suppose that G is internally indecomposable. Then $G^n \rtimes \mathfrak{S}_n$ is internally indecomposable, but not 2-sn-internally indecomposable.*

Proof. For each $i \in \{1, \ldots, n\}$, write

- G_i for the i-th component of G^n;
- $\iota_i : G \hookrightarrow G^n$ for the composite of the natural isomorphism $G \xrightarrow{\sim} G_i$ and the natural inclusion homomorphism $G_i \hookrightarrow G^n$;
- $p_i : G^n \twoheadrightarrow G$ for the i-th projection homomorphism.

For each subgroup $H \subseteq G$ of G, we identify H^n with $\iota_1(H) \times \cdots \times \iota_n(H) \subseteq G^n$.

First, let us observe that, for any nontrivial subgroup $H \subseteq G$, it holds that $Z_{G^n \rtimes \mathfrak{S}_n}(H^n) \subseteq G^n$, hence $Z_{G^n \rtimes \mathfrak{S}_n}(H^n) = Z_{G^n}(H^n) = (Z_G(H))^n$. Indeed, let $(g, \sigma) = ((g_1, \ldots, g_n), \sigma) \in Z_{G^n \rtimes \mathfrak{S}_n}(H^n)$. Then, for any $h = (h_1, \ldots, h_n) \in H^n$, it holds that $(g, \sigma) = h^{-1}(g, \sigma)h = (h^{-1} g \sigma(h), \sigma)$. This implies that for each $i \in \{1, \ldots, n\}$, it holds that $g_i^{-1} h_i g_i = h_{\sigma^{-1}(i)}$. Since H is nontrivial, to satisfy this equality for any $h \in H^n$, it must be $\sigma = \mathrm{id}$. Thus, we conclude that $Z_{G^n \rtimes \mathfrak{S}_n}(H^n) \subseteq G^n$.

Next, we verify assertion (i). Since N and G^n are normal in $G^n \rtimes \mathfrak{S}_n$, it holds that $[N, G^n] \subseteq N \cap G^n = \{1\}$. Thus, it holds that $N \subseteq Z_{G^n \rtimes \mathfrak{S}_n}(G^n)$. In particular, it follows from the above observation that $N \subseteq G^n$, which implies that $N = N \cap G^n = \{1\}$. This completes the proof of assertion (i).

Finally, we verify assertion (ii). Since the normal subgroup $G^n \subseteq G^n \rtimes \mathfrak{S}_n$ is not internally indecomposable, $G^n \rtimes \mathfrak{S}_n$ is not 2-sn-internally indecomposable. Thus, it suffices to prove that, for any nontrivial normal subgroup $N \subseteq G^n \rtimes \mathfrak{S}_n$, it holds that $Z_{G^n \rtimes \mathfrak{S}_n}(N) = \{1\}$ [cf. Proposition 1.7].

Now it follows from assertion (i) that there exists $i \in \{1, \ldots, n\}$ such that $p_i(N \cap G^n) \neq \{1\}$. Write $H \stackrel{\text{def}}{=} [G, p_i(N \cap G^n)]$. Then, since G is a nontrivial internally indecomposable group, hence center-free, it holds that $H \neq \{1\}$. Moreover, since $p_i(N \cap G^n) \subseteq G$ is a normal subgroup of G, H is also normal in G, which implies that $Z_G(H) = \{1\}$ [cf. Proposition 1.7]. In particular, it follows from the above observation that $Z_{G^n \rtimes \mathfrak{S}_n}(H^n) = (Z_G(H))^n = \{1\}$.

On the other hand, since G_i commutes with G_j for any $j \in \{1, \ldots, n\}$ such that $j \neq i$, it is clear that $\iota_i(H) = [G_i, \iota_i(p_i(N \cap G^n))] = [G_i, N \cap G^n] \subseteq N$. Moreover, for $j \in \{1, \ldots, n\}$, if we take $\tau \in \mathfrak{S}_n$ such that $\tau^{-1}(i) = j$, then it holds that $\iota_j(H) = (1, \tau)^{-1} \iota_i(H)(1, \tau) \subseteq N$. Thus, we obtain that $H^n \subseteq N$, which implies that $Z_{G^n \rtimes \mathfrak{S}_n}(N) \subseteq Z_{G^n \rtimes \mathfrak{S}_n}(H^n) = \{1\}$. This completes the proof of assertion (ii), hence also of Proposition 1.9. □

Proposition 1.10. *Let G be a Hausdorff topological group; $H \subseteq G$ a subgroup; n a nonnegative integer. If $\overline{H} \subseteq G$ is n-sn-internally indecomposable, then so is H.*

Proof. Suppose that $\overline{H}$ is n-sn-internally indecomposable. Let $S \subseteq H$ be a nontrivial n-subnormal subgroup of H. Then it follows from Remark 1.1.1 that $\overline{S}$ is n-subnormal in $\overline{H}$, which implies that $Z_{\overline{H}}(\overline{S}) = \{1\}$. Now it follows from Lemma 1.6, (ii), that $Z_H(S) = Z_{\overline{H}}(S) \cap H = Z_{\overline{H}}(\overline{S}) \cap H = \{1\}$. This completes the proof of Proposition 1.10. □

Proposition 1.11. *Let G be a group; n a positive integer; $\{H_\lambda\}_{\lambda \in \Lambda}$ a set of n-subnormal subgroups of G. Then the following hold:*

(i) *$\bigcap_{\lambda \in \Lambda} H_\lambda$ is n-subnormal in G.*
(ii) *Suppose that G is n-sn-internally indecomposable, that Λ is finite, and that for each $\lambda \in \Lambda$, H_λ is nontrivial. Then $\bigcap_{\lambda \in \Lambda} H_\lambda$ is nontrivial.*

Proof. For each $\lambda \in \Lambda$, there exist subgroups $H_{\lambda,0} = G, H_{\lambda,1}, \ldots, H_{\lambda,n-1}, H_{\lambda,n} = H_\lambda$ of G such that $H_{\lambda,i}$ is normal in $H_{\lambda,i-1}$ for each $i \in \{1, \ldots, n\}$. Then, since $\bigcap_{\lambda \in \Lambda} H_{\lambda,i}$ is a normal subgroup of $\bigcap_{\lambda \in \Lambda} H_{\lambda,i-1}$, assertion (i) is clear. Next, we verify assertion (ii). By assertion (i), we may assume that Λ consists of two elements λ, μ. For each $i \in \{0, \ldots, n\}$, write $H_i \stackrel{\text{def}}{=} H_{\lambda,i}$; $S_i \stackrel{\text{def}}{=} H_{\mu,i}$. We prove that for $i, j \in \{0, \ldots, n\}$, it holds that $H_i \cap S_j \neq \{1\}$ by induction on $i + j$. The case where $i + j \leq 1$ is clear.

Now suppose that $i + j \geq 2$, and that the induction hypothesis is in force. We may assume that $i, j \neq 0$. Then it follows from the induction hypothesis that $H_{i-1} \cap S_j$ and $H_i \cap S_{j-1}$ are nontrivial. Moreover, since $H_{i-1} \cap S_j$ and $H_i \cap S_{j-1}$ are normal subgroups of $H_{i-1} \cap S_{j-1}$, it holds that $[H_{i-1} \cap S_j, H_i \cap S_{j-1}] \subseteq (H_{i-1} \cap S_j) \cap (H_i \cap S_{j-1}) = H_i \cap S_j$. Thus, if $H_i \cap S_j = \{1\}$, then we obtain a normally decomposable subgroup $(H_{i-1} \cap S_j) \times (H_i \cap S_{j-1})$ of $H_{i-1} \cap S_{j-1}$.

On the other hand, it follows from assertion (i) that $H_{i-1} \cap S_{j-1}$ is $\max\{i-1, j-1\}$-subnormal in G, hence $(n-1)$-subnormal in G. Thus, since G is n-sn-internally indecomposable, we obtain a contradiction. Therefore, we conclude that $H_i \cap S_j \neq \{1\}$. This completes the proof of assertion (ii), hence also of Proposition 1.11. ☐

Proposition 1.12. *Let G be a profinite group; $H \subseteq G$ an open subgroup; n a positive integer. Suppose that any open subgroup of G has no nontrivial finite normal subgroup [e.g. the case where G is slim [cf. [13], Lemma 1.3]] and that H is strongly n-sn-internally indecomposable. Then G is strongly n-sn-internally indecomposable.*

Proof. We prove Proposition 1.12 by induction on n. The case where $n = 1$ follows from the proof of [14], Proposition 1.7. Now suppose that $n \geq 2$, and that the induction hypothesis is in force. It suffices to prove that G is n-sn-internally indecomposable. Let $S \subseteq G$ be an $(n-1)$-subnormal closed subgroup of G. Note that, since H is strongly n-sn-internally indecomposable, hence strongly $(n-1)$-sn-internally indecomposable, it follows from the induction hypothesis that G is strongly $(n-1)$-sn-internally indecomposable. In particular, it follows from Corollary 1.3 that S is slim. Moreover, since $S \cap H$ is $(n-1)$-subnormal in H, it holds that $S \cap H$ is strongly internally indecomposable. Thus, it follows from [14], Proposition 1.7, that S is [strongly] internally indecomposable, which implies that G is n-sn-internally indecomposable. This completes the proof of Proposition 1.12. ☐

Proposition 1.13. *Let G be a group (respectively, a profinite group); n a positive integer; $\{G_i\}_{i \in I}$ a directed subset of the set of subgroups (respectively, closed subgroups) of G [where $j \geq i \Leftrightarrow G_i \subseteq G_j$] such that $G = \bigcup_{i \in I} G_i$. If for each $i \in I$, G_i is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable), then so is G.*

Proof. The profinite group case follows immediately from the general group case. We verify the general group case. Suppose that G_i is n-sn-internally indecomposable for each $i \in I$. Let $H \subseteq G$ be a nontrivial n-subnormal subgroup. Fix an element $i \in I$ such that $H \cap G_i \neq \{1\}$ and write $I_i \stackrel{\text{def}}{=} \{j \in I \,|\, j \geq i\}$. Then for each $j \in I_i$, since $H \cap G_j$ is a nontrivial n-subnormal subgroup of G_j, it holds that $Z_G(H) \cap G_j \subseteq Z_{G_j}(H \cap G_j) = \{1\}$. Thus, we conclude that $Z_G(H) = \bigcup_{j \in I_i} (Z_G(H) \cap G_j) = \{1\}$. This completes the proof of Proposition 1.13. ☐

Proposition 1.14. *Let G be a group (respectively, a profinite group); n a positive integer; $\{G_i\}_{i \in I}$ a directed subset of the set of normal subgroups (respectively, normal closed subgroups) of G [where $j \geq i \Leftrightarrow G_j \subseteq G_i$] such that the natural homomorphism $G \to \varprojlim_{i \in I} G/G_i$ is an isomorphism. If for each $i \in I$, G/G_i is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable), then so is G.*

Proof. The profinite group case follows immediately from the general group case. We verify the general group case. Suppose that G/G_i is n-sn-internally indecomposable for each $i \in I$. Write $\phi_i : G \twoheadrightarrow G/G_i$ for the natural surjection. Let $H \subseteq G$ be a nontrivial n-subnormal subgroup. Fix an element $i \in I$ such that $\phi_i(H) \neq \{1\}$ and write $I_i \stackrel{\text{def}}{=} \{j \in I \,|\, j \geq i\}$. Then for each $j \in I_i$, since $\phi_j(H)$ is a nontrivial n-subnormal subgroup of G/G_j, it holds that $\phi_j(Z_G(H)) \subseteq Z_{G/G_j}(\phi_j(H)) = \{1\}$.

Thus, we conclude that $Z_G(H) \subseteq \bigcap_{j \in I_i} G_j = \{1\}$. This completes the proof of Proposition 1.14. $\square$

Lemma 1.15. *Let n be a positive integer; G an n-sn-internally indecomposable group; $S \subseteq G$ a nontrivial n-subnormal subgroup; $H \subseteq G$ a subgroup containing S; $\alpha : H \to G$ a homomorphism. Suppose that for any $h \in S$, it holds that $\alpha(h) = h$. Then for any $g \in H$, it holds that $\alpha(g) = g$.*

Proof. There exist subgroups $S_0 = G, S_1, \ldots, S_{n-1}, S_n = S$ of G such that S_i is normal in S_{i-1} for each $i \in \{1, \ldots, n\}$. It suffices to prove that for $i \in \{1, \ldots, n\}$, if $\alpha(h) = h$ for any $h \in S_i \cap H$, then for any $g \in S_{i-1} \cap H$, it holds that $\alpha(g) = g$.

Suppose that for any $h \in S_i \cap H$, it holds that $\alpha(h) = h$. Let $g \in S_{i-1} \cap H$. Then since S_i is normal in S_{i-1}, for any $h \in S_i \cap H$, it holds that $g^{-1}hg \in S_i \cap H$, hence $\alpha(g^{-1}hg) = g^{-1}hg$. Thus, we obtain that $\alpha(g)g^{-1}h(\alpha(g)g^{-1})^{-1} = \alpha(g)\alpha(g^{-1}hg)\alpha(g)^{-1} = \alpha(h) = h$. This implies that $\alpha(g)g^{-1} \in Z_G(S_i \cap H) \subseteq Z_G(S) = \{1\}$. This completes the proof of Lemma 1.15. $\square$

Proposition 1.16. *Let $1 \to G_1 \to G \to G_2 \to 1$ be an exact sequence of groups (respectively, profinite groups); n a positive integer. Write $\rho : G_2 \to \mathrm{Out}(G_1)$ for the outer representation associated to this exact sequence. Suppose that*

- *G_1 is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable);*
- *G_2 is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable);*
- *ρ is injective.*

Then G is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable).

Proof. Write φ for the surjective homomorphism $G \twoheadrightarrow G_2$ appearing in the statement. It follows from [12], Lemma 1.7, (i), that it suffices to prove the general group case. Let $H \subseteq G$ be an $(n-1)$-subnormal subgroup. If $H \cap G_1 = \{1\}$, then, since $H \xrightarrow{\sim} \varphi(H) \subseteq G_2$ is $(n-1)$-subnormal in G_2, it holds that H is internally indecomposable.

If $H \cap G_1 \neq \{1\}$, then let $h \in H$ be an element such that $\varphi(h)$ is in the kernel of the natural outer representation $\varphi(H) \to \mathrm{Out}(H \cap G_1)$. Then there exists an element $k \in H \cap G_1$ such that for any $g \in H \cap G_1$, it holds that $(kh)g(kh)^{-1} = g$. Thus, since $H \cap G_1 \subseteq G_1$ is $(n-1)$-subnormal in G_1, it follows from Lemma 1.15 that $(kh)g(kh)^{-1} = g$ for any $g \in G_1$. This implies that $\varphi(h) = \varphi(kh) \in \ker \rho = \{1\}$. Thus, the outer representation $\varphi(H) \to \mathrm{Out}(H \cap G_1)$ is injective. Moreover, since $H \cap G_1 \subseteq G_1$ and $\varphi(H) \subseteq G_2$ are $(n-1)$-subnormal in G_1, G_2, respectively, it holds that $H \cap G_1$ and $\varphi(H)$ are internally indecomposable. Thus, it follows from [the proof of] [14], Proposition 1.11, (i), that H is internally indecomposable. This completes the proof of Proposition 1.16. $\square$

Proposition 1.17. *Let $1 \to G_1 \to G \to G_2 \to 1$ be an exact sequence of groups (respectively, profinite groups); n a positive integer. Suppose that*

- *G_1 is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable);*
- *G_2 is abelian;*
- *G is center-free (respectively, slim).*

Then G is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable).

Proof. To verify Proposition 1.17, it suffices to prove the general group case. Let $H \subseteq G$ be an $(n-1)$-subnormal subgroup. Then there exist subgroups $H_0 = G, H_1, \ldots, H_{n-2}, H_{n-1} = H$ of G such that H_i is normal in H_{i-1} for each $i \in \{1, \ldots, n-1\}$. We prove that for $i \in \{0, \ldots, n-1\}$, H_i is internally indecomposable by induction on n. If $i = 0$, then this follows from [the proof of] [14], Proposition 1.11, (ii).

Now suppose that $i \geq 1$, and that the induction hypothesis is in force. Then H_{i-1} is internally indecomposable by the induction hypothesis, which implies that H_i is center-free. Moreover, since G_1 is n-sn-internally indecomposable, $H_i \cap G_1$ is internally indecomposable. Furthermore, $H_i/(H_i \cap G_1)$ is isomorphic to a subgroup of G_2, hence is abelian. Thus, it follows from [the proof of] [14], Proposition 1.11, (ii), that H_i is internally indecomposable. This completes the proof of Proposition 1.17. □

Remark 1.17.1. Let $1 \to G_1 \to G \to G_2 \to 1$ be an exact sequence of groups (respectively, profinite groups); n a positive integer. Suppose that G_1 is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable). Then G is center-free (respectively, slim) if the outer representation $G_2 \to \mathrm{Out}(G_1)$ is injective.

Corollary 1.18. *Let G be a group; m, n positive integers; $H_0 = G, H_1, \ldots, H_m$ closed subgroups of G such that H_i is normal in H_{i-1} for each $i \in \{1, \ldots, m\}$. Suppose that*

- *H_m is nontrivial and n-sn-internally indecomposable;*
- *for each $i \in \{1, \ldots, m\}$, H_{i-1}/H_i is n-sn-internally indecomposable or abelian;*
- *H_m is n-subnormal in G [e.g. the case where $m \leq n$].*

Then the following conditions are equivalent:

(1) *G is n-sn-internally indecomposable.*
(2) *It holds that $Z_G(H_m) = \{1\}$.*
(3) *For each $i \in \{1, \ldots, m\}$, it holds that $Z_{H_{i-1}}(H_i) = \{1\}$.*
(4) *For each $i \in \{1, \ldots, m\}$, the natural outer representation $H_{i-1}/H_i \to \mathrm{Out}(H_i)$ is injective.*

Moreover, if further suppose that $G, H_1, \ldots, H_m$ are profinite, that H_m is strongly n-sn-internally indecomposable, and that for each $i \in \{1, \ldots, m\}$, H_{i-1}/H_i is strongly n-sn-internally indecomposable or abelian, then the above conditions are equivalent to the following condition:

(5) *G is strongly n-sn-internally indecomposable.*

Proof. The second assertion follows from the first assertion and Propositions 1.16, 1.17; Remark 1.17.1. We verify the first assertion. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate. The implication (4) $\Rightarrow$ (1) follows from Propositions 1.16, 1.17; Remark 1.17.1. This completes the proof of Corollary 1.18. □

2. Elasticity of subnormal subgroups

In the present section, we discuss generalities on *elasticity* of subnormal subgroups of profinite groups [cf. Definition 2.1].

Definition 2.1. Let G be a profinite group; n a positive integer.

(i) We shall say that G is *n-sn-quasielastic* (respectively, *n-sn-elastic*) if every topologically finitely generated n-subnormal closed subgroup of G (respectively, of an open subgroup of G) is trivial or open. We shall say that G is *quasielastic* (respectively, *elastic*) if G is 1-sn-quasielastic (respectively, 1-sn-elastic).

(ii) We shall say that G is *sn-quasielastic* (respectively, *sn-elastic*) if G is m-sn-quasielastic (respectively, m-sn-elastic) for any positive integer m.

(iii) We shall say that G is *very quasielastic* (respectively, *very n-sn-quasielastic*; *very n-sn-elastic*; *very sn-quasielastic*; *very sn-elastic*) if G is quasielastic (respectively, n-sn-quasielastic; n-sn-elastic; sn-quasielastic; sn-elastic), but not topologically finitely generated.

Remark 2.1.1. Although we define "elasticity" here as a special case of n-sn-elasticity, the definition of "elasticity" is exactly the same as in common use [cf. [21], Definition 1.1, (ii)].

Lemma 2.2. *Let G be a profinite group; n a positive integer. Then the following hold:*

(i) *The following conditions are equivalent:*
  (1) *G is n-sn-quasielastic.*
  (2) *Every $(n-1)$-subnormal closed subgroup of G is quasielastic.*

(ii) *Suppose that G is nontrivial. Then the following conditions are equivalent:*
  (1) *G is very n-sn-quasielastic.*
  (2) *Every nontrivial $(n-1)$-subnormal closed subgroup of G is very quasielastic.*
  (3) *Every topologically finitely generated n-subnormal closed subgroup of G is trivial.*

Proof. First, we verify assertion (i). The implication (1) $\Rightarrow$ (2) is clear. We verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. Let $H \subseteq G$ be a nontrivial topologically finitely generated n-subnormal closed subgroup. Then there exist closed subgroups $H_0 = G, H_1, \ldots, H_{n-1}, H_n = H$ of G such that H_i is normal in H_{i-1} for each $i \in \{1, \ldots, n\}$. Now since H_{n-1} is quasielastic, $H_n \subseteq H_{n-1}$ is open in H_{n-1}. In particular, H_{n-1} is topologically finitely generated. Thus, by applying the above argument inductively, we observe that for each $i \in \{1, \ldots, n\}$, H_i is open in H_{i-1}. Therefore, we conclude that H is open in G. This completes the proof of assertion (i).

Next, we verify assertion (ii). The implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear. The implication (2) $\Rightarrow$ (1) follows from assertion (i). This completes the proof of assertion (ii), hence also of Lemma 2.2. ☐

Lemma 2.3. *Let G be a profinite group; $H \subseteq G$ an open subgroup; n a positive integer. Suppose that G has no nontrivial finite n-subnormal subgroup [e.g. the case where every $(n-1)$-subnormal closed subgroup of G is slim [cf. [13], Lemma 1.3]]. Then if H is n-sn-quasielastic (respectively, very n-sn-quasielastic), then so is G.*

Proof. If $S \subseteq G$ is a nontrivial topologically finitely generated n-subnormal closed subgroup, then $H \cap S$ is a nontrivial topologically finitely generated n-subnormal closed subgroup of H. This implies that $H \cap S$ is open in H, hence also in G. Thus, we conclude that S is open in G. This completes the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let G be a profinite group; n a positive integer; $\{G_i\}_{i \in I}$ a directed subset of the set of closed subgroups of G [where $j \geq i \Leftrightarrow G_j \subseteq G_i$] such that the natural homomorphism $G \to \varprojlim_{i \in I} G/G_i$ is an isomorphism. If for each $i \in I$, G/G_i is very n-sn-quasielastic (respectively, very n-sn-elastic), then so is G.*

Proof. To verify Lemma 2.4, it suffices to prove the case where G/G_i is very n-sn-quasielastic. For each $i \in I$, write $\phi_i : G \twoheadrightarrow G/G_i$ for the natural surjection. Let $H \subseteq G$ be a topologically finitely generated n-subnormal closed subgroup. Then it follows from Lemma 2.2, (ii), that for each $i \in I$, $\phi_i(H) = \{1\}$. Thus, it holds that $H \subseteq \bigcap_{i \in I} G_i = \{1\}$. This implies that G is very n-sn-quasielastic [cf. Lemma 2.2, (ii)]. This completes the proof of Lemma 2.4. $\square$

Lemma 2.5. *Let p be a prime number; G a pro-p group such that every nontrivial abelian closed subgroup of G is isomorphic to $\mathbb{Z}_p$. Then the following hold:*

(i) *Let $H \subseteq G$ be a nontrivial closed subgroup such that $Z(H) = \{1\}$. Then $Z_G(H) = \{1\}$.*

(ii) *Let n be a positive integer. Suppose that*
- *every open subgroup of G is not isomorphic to $\mathbb{Z}_p$;*
- *G is n-sn-quasielastic (respectively, n-sn-elastic).*

Then G is n-sn-internally indecomposable (respectively, strongly n-sn-internally indecomposable).

Proof. First, we verify assertion (i). Let us observe that $H \cap Z_G(H) = Z(H) = \{1\}$. Let $x \in H \setminus \{1\}$ and $y \in Z_G(H)$. Then the [abelian] closed subgroup of G topologically generated by $x, y \in G$ is isomorphic to $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is indecomposable, we obtain that $y = 1$. Thus, it holds that $Z_G(H) = \{1\}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). It suffices to prove the case where G is n-sn-quasielastic. Let $H \subseteq G$ be a nontrivial n-subnormal closed subgroup of G. Then, since $Z(H) \subseteq H$ is characteristic in H, $Z(H)$ is an abelian n-subnormal closed subgroup of G. In particular, $Z(H)$ is isomorphic to $\mathbb{Z}_p$. Thus, we conclude from the assumptions on G in the statement of assertion (ii) that $Z(H) = \{1\}$, hence from assertion (i) that $Z_G(H) = \{1\}$. This completes the proof of assertion (ii), hence also of Lemma 2.5. $\square$

Definition 2.6 (cf. [3], §17.3; [22], Definition 1.1, (i), (ii))**.** Let $\mathcal{C}$ be a family of finite groups containing the trivial group; Σ a set of prime numbers.

(i) We shall refer to a [finite] group belonging to $\mathcal{C}$ as a *$\mathcal{C}$-group*. We shall refer to a finite group as a *Σ-group* if every prime factor of its order belongs to Σ.

(ii) We shall refer to $\mathcal{C}$ as a *full-formation* if $\mathcal{C}$ is closed under taking quotients, subgroups, and extensions.

(iii) We shall write $\Sigma_{\mathcal{C}}$ for the set of prime numbers l such that $\mathbb{Z}/l\mathbb{Z}$ is a $\mathcal{C}$-group.

Definition 2.7 (cf. [3], Definition 17.3.2; [22], Definition 1.1, (iii))**.** Let G be a profinite group; $\mathcal{C}$ a full-formation; Σ a nonempty set of prime numbers; l a prime number.

(i) We shall write $G^{\mathcal{C}}$ for the maximal pro-$\mathcal{C}$ quotient of G. If $\mathcal{C}$ is the family of all Σ-groups, then we shall also write $G^{\Sigma} \stackrel{\text{def}}{=} G^{\mathcal{C}}$. Moreover, we shall also write $G^{l} \stackrel{\text{def}}{=} G^{\{l\}}$.

(ii) Let Q be a quotient of G [in the category of profinite groups]; $N \subseteq G$ a normal open subgroup. If the kernel of the surjection $G \twoheadrightarrow Q$ coincides with the kernel of the natural surjection $N \twoheadrightarrow N^{\mathcal{C}}$ (respectively, $N \twoheadrightarrow N^{\Sigma}$; $N \twoheadrightarrow N^{l}$), then we shall say that Q is the *almost pro-$\mathcal{C}$-maximal quotient* (respectively, *almost pro-Σ-maximal quotient*; *almost pro-l-maximal quotient*) of G associated to N.

(iii) Let Q be a quotient of G. Then we shall say that Q is an *almost pro-$\mathcal{C}$-maximal quotient* (respectively, *almost pro-Σ-maximal quotient*; *almost pro-l-maximal quotient*) of G if it is the almost pro-$\mathcal{C}$-maximal quotient (respectively, almost pro-Σ-maximal quotient; almost pro-l-maximal quotient) of G associated to N for some normal open subgroup $N \subseteq G$.

Lemma 2.8. *Let G be a profinite group; p a prime number; n a positive integer. Suppose that for any positive integer m, there exists a positive integer d_m such that any open subgroup U of G of index $\geq d_m$ satisfies the following conditions:*

- *any almost pro-p-maximal quotient of U is n-sn-quasielastic;*
- *there is no open subgroup of U^p topologically generated by m elements.*

Then G is n-sn-elastic.

Proof. Let $V \subseteq G$ be an open subgroup of G; $H \subseteq V$ a topologically finitely generated n-subnormal closed subgroup of V of infinite index. Write $m \stackrel{\text{def}}{=} r(H)$. Let $U \subseteq V$ be an open subgroup of V of index $\geq d_m$ containing H. Then it follows from the second assumption on d_m that the image of H in U^p is not open, which implies that the image of H in any almost pro-p-maximal quotient of U is not open. Thus, we conclude from the first assumption on d_m that the image of H in any almost pro-p-maximal quotient of U is trivial. Therefore, we conclude that H is trivial. This completes the proof of Lemma 2.8. ☐

3. Subnormal subgroups of groups in anabelian geometry

In the present section, we apply the generalities discussed in the previous sections to prove the sn-internal indecomposability and the sn-elasticity of various groups appearing in anabelian geometry.

In the present section, let p be a prime number. First, we consider properties of free pro-$\mathcal{C}$ groups and Demuškin groups.

Lemma 3.1. *Let G be a profinite group; $\mathcal{C}$ a full-formation; $N \subseteq G$ a normal open subgroup; $\{G_i\}_{i \in I}$ a directed subset of the set of normal closed subgroups of G [where $j \geq i \Leftrightarrow G_j \subseteq G_i$] such that the natural homomorphism $G \to \varprojlim_{i \in I} G/G_i$ is an isomorphism. Write Q for the almost pro-$\mathcal{C}$-maximal quotient of G associated to N. Moreover, for each $i \in I$, write Q_i for the almost pro-$\mathcal{C}$-maximal quotient of G/G_i associated to the image of N in G/G_i. Then the natural homomorphism $Q \to \varprojlim_{i \in I} Q_i$ is an isomorphism.*

Proof. Write $K \stackrel{\mathrm{def}}{=} \ker(G \twoheadrightarrow Q) = \ker(N \twoheadrightarrow N^{\mathcal{C}})$; $K_i \stackrel{\mathrm{def}}{=} \ker(G/G_i \twoheadrightarrow Q_i)$. Then for each $i \in I$, we obtain a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_i \arrow[r] & G/G_i \arrow[r] & Q_i \arrow[r] & 1,
\end{tikzcd}$$

where the horizontal sequences are exact and the vertical arrows are surjective [cf. [26], Lemma 3.4.1, (b)]. By taking the inverse limit, we obtain a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \varprojlim_{i \in I} K_i \arrow[r] & \varprojlim_{i \in I} G/G_i \arrow[r] & \varprojlim_{i \in I} Q_i \arrow[r] & 1,
\end{tikzcd}$$

where the horizontal sequences are exact [cf. [26], Proposition 2.2.4] and the vertical arrows are surjective [cf. [26], Lemma 1.1.5]. Thus, since $G \to \varprojlim_{i \in I} G/G_i$ is an isomorphism, we conclude that the natural homomorphism $Q \to \varprojlim_{i \in I} Q_i$ is an isomorphism. This completes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Let $\mathcal{C}$ be a nontrivial full-formation. Then any almost pro-$\mathcal{C}$-maximal quotient of a free profinite group of [possibly infinite] rank ≥ 2 is slim.*

Proof. This follows from Lemma 3.1; [26], Corollary 3.3.10, (b); [22], Proposition 1.4. $\square$

Theorem 3.3. *Let $\mathcal{C}$ be a nontrivial full-formation. Then any almost pro-$\mathcal{C}$-maximal quotient of a free profinite group of [possibly infinite] rank ≥ 2 is strongly sn-internally indecomposable and sn-elastic.*

Proof. First, we claim the following:

> Claim 3.3.A: Every free pro-p group of rank ≥ 2 is strongly sn-internally indecomposable and sn-elastic.

Indeed, let F be a free pro-p group of rank ≥ 2. Then it follows from [26], Theorem 3.6.2, (b); [26], Corollary 7.7.5; [26], Theorem 8.6.6, that every nontrivial normal closed subgroup of an open subgroup of F, hence also every nontrivial subnormal closed subgroup of an open subgroup of F, is a free pro-p group of rank ≥ 2. Thus, it follows from Lemma 2.2, (i); [14], Proposition 1.5; [26], Theorem 8.6.6, that F is strongly sn-internally indecomposable and sn-elastic. This completes the proof of Claim 3.3.A.

Now let G be an almost pro-$\mathcal{C}$-maximal quotient of a free profinite group of rank ≥ 2. We may assume that $p \in \Sigma_{\mathcal{C}}$. Next, we claim the following:

> Claim 3.3.B: G is strongly sn-internally indecomposable.

Indeed, in light of Proposition 1.14, we may assume that G is an almost pro-p-maximal quotient of a free profinite group of rank ≥ 2. Then Claim 3.3.B follows from Proposition 1.12; Lemma 3.2; Claim 3.3.A.

To complete the proof of Theorem 3.3, it suffices to prove that G is sn-elastic. In light of Claim 3.3.B; Lemmas 2.3, we may assume that G is a free pro-$\mathcal{C}$ group of rank ≥ 2. Then it follows from Claims 3.3.A, 3.3.B; Lemmas 2.3, 2.8; [26], Theorem 3.6.2, that G is sn-elastic. This completes the proof of Theorem 3.3. □

Remark 3.3.1. Theorem 3.3 may be regarded as a generalization of [14], Proposition 1.5 [which is a consequence of [26], Proposition 8.7.8]. Note that by applying Lemma 2.5, (ii), instead of [14], Proposition 1.5, in the above proof, we obtain an alternative proof of [14], Proposition 1.5.

Corollary 3.4. *Every free pro-p group is sn-elastic.*

Proof. Since any nontrivial closed subgroup of $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p$, Corollary 3.4 follows immediately from Theorem 3.3. □

Definition 3.5 ([25], Definition 3.9.9)**.** Let G be a pro-p group. Then we shall say that G is a *Demuškin group* if the following conditions are satisfied:

- $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$;
- $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$;
- the cup product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is nondegenerate.

Proposition 3.6. *Let n be an integer such that $n \geq 2$; G an infinite pro-p Demuškin group of rank n. Then the following hold:*

(i) *Every open subgroup $U \subseteq G$ is a pro-p Demuškin group of rank $2 + [G : U](n-2)$.*
(ii) *Every nontrivial closed subgroup of infinite index of G is a free pro-p group.*
(iii) *Suppose that G is not isomorphic to $\mathbb{Z}_p^2$. Then G is slim and strongly indecomposable.*

Proof. Assertion (i) follows from [25], Theorem 3.9.15. Assertion (ii) follows from [25], Chapter III, §7, Exercise 3, (ii).

We verify assertion (iii). First, we verify that G is slim. In light of assertion (i) and [11], Lemma 2.2, (i), it suffices to prove that $Z(G) = \{1\}$. Suppose that $Z(G) \neq \{1\}$. Let us observe that any nontrivial abelian closed subgroup of G is isomorphic to $\mathbb{Z}_p$ [cf. assertions (i), (ii); [11], Lemma 2.2, (i)]. In particular, it holds that $Z(G) \cong \mathbb{Z}_p$. Thus, it follows from [25], Theorem 3.3.9, that $G/Z(G)$ is of virtual p-cohomological dimension 1, which implies that $G/Z(G)$ has a free pro-p open subgroup $V \subseteq G/Z(G)$. Then the closed subgroup of G topologically generated by $Z(G)$ and a lifting of a nontrivial element of V in G is isomorphic to $\mathbb{Z}_p^2$, which contradicts the above observation. This completes the proof of the slimness of G.

Next, we verify that G is strongly indecomposable. In light of assertion (i) and [11], Lemma 2.2, (i), it suffices to prove that G is indecomposable. Suppose that $G = H_1 \times H_2$ and $H_1 \neq \{1\}$. Then, since $Z(H_1) \times Z(H_2) = Z(G) = \{1\}$, we obtain that $Z(H_1) = \{1\}$. Thus, it follows from Lemma 2.5, (i), together with the above observation, that $Z_G(H_1) = \{1\}$. Since $H_2 \subseteq Z_G(H_1)$, we conclude that $H_2 = \{1\}$. This completes the proof of assertion (iii), hence also of Proposition 3.6. □

Theorem 3.7. *Every pro-p Demuškin group of rank ≥ 3 is strongly sn-internally indecomposable and sn-elastic.*

Proof. Let G be a pro-p Demuškin group of rank ≥ 3. Note that G is infinite [cf. [25], Proposition 3.9.10]. In light of Lemma 2.5, (ii); Proposition 3.6, (i), (ii), it suffices to prove that G is sn-quasielastic. Let $H \subseteq G$ be a topologically finitely generated subnormal closed subgroup of infinite index. Take an open subgroup $U \subseteq G$ of G containing H such that $[G : U] \geq r(H) + 1$. Then it follows from Proposition 3.6, (i), that the inequality $r(U) \geq r(H)+3$ holds. Since U^{ab} contains a subgroup isomorphic to $\mathbb{Z}_p^{r(U)-1}$ [cf. the discussion preceding [25], Theorem 3.9.11], there exists a closed subgroup $F \subseteq U$ of U of infinite index such that H is a closed subgroup of F of infinite index. Now it follows from Corollary 3.4; Proposition 3.6, (ii), that H is trivial. Thus, we conclude that G is sn-quasielastic. This completes the proof of Theorem 3.7. □

Next, we prove the sn-internal indecomposability and the sn-elasticity of [almost pro-$\mathcal{C}$-maximal quotients of] the absolute Galois groups of various fields.

Theorem 3.8. *Let K be a Henselian discrete valuation field [cf. [3], §11.5] of residue characteristic p; $\mathcal{C}$ a full-formation such that $p \in \Sigma_{\mathcal{C}}$. Then any almost pro-$\mathcal{C}$-maximal quotient of G_K is strongly sn-internally indecomposable and sn-elastic.*

Proof. If K is of characteristic p, then it follows from the proof of [14], Theorem 2.3, that any almost pro-p-maximal quotient of G_K is slim and has an open subgroup which is a free pro-p group of infinite rank. Thus, it follows from Propositions 1.12, 1.14; Theorem 3.3, that any almost pro-$\mathcal{C}$-maximal quotient of G_K is strongly sn-internally indecomposable. Moreover, it follows from Lemmas 2.3, 2.4; Theorem 3.3, that any almost pro-$\mathcal{C}$-maximal quotient of G_K is very sn-elastic. This completes the proof of Theorem 3.8 in the case where K is of characteristic p.

In the remainder of the proof of Theorem 3.8, we assume that K is a mixed characteristic Henselian discrete valuation field [of residue characteristic p]. First, we claim the following:

> Claim 3.8.A: Suppose that the residue field of K is infinite. Then G_K^p is very sn-elastic.

Indeed, it suffices to prove that G_K^p is very sn-quasielastic. Let $H \subseteq G_K^p$ be a topologically finitely generated subnormal closed subgroup. Then it follows from [15], Proposition 3.3, that there exists a closed subgroup $Q \subseteq G_K^p$ of infinite index such that H is a closed subgroup of Q of infinite index, and that Q is a free pro-p group. Then it follows from Corollary 3.4 that H is trivial. Thus, it follows from Lemma 2.2, (ii), that G_K^p is very sn-quasielastic. This completes the proof of Claim 3.8.A.

Next, we claim the following:

> Claim 3.8.B: Suppose that the residue field of K is infinite. Then any almost pro-$\mathcal{C}$-maximal quotient of G_K is strongly sn-internally indecomposable.

Indeed, it follows from Lemma 2.5, (ii); Claim 3.8.A; [15], Corollary 3.6, that G_K^p is sn-internally indecomposable. Thus, Claim 3.8.B follows from Propositions 1.12, 1.14; [15], Theorem 4.3.

In light of Claim 3.8.B, to prove Theorem 3.8 in the case where the residue field of K is infinite, it suffices to prove that any almost pro-$\mathcal{C}$-maximal quotient of G_K is [very] sn-elastic. Moreover, in light of Claim 3.8.B; Lemmas 2.3, 2.4, it suffices to prove that G_K^p is very sn-elastic. This is nothing but Claim 3.8.A. This completes the proof of Theorem 3.8 in the case where the residue field of K is infinite.

To complete the proof of Theorem 3.8, we may assume that the residue field of K is finite. Moreover, by [13], Lemma 3.1, we may assume that K is a p-adic local field. Then it follows from Theorems 3.3, 3.7; [25], Theorem 7.5.11, that G_K^p is sn-internally indecomposable and sn-elastic. Moreover, it follows from Propositions 1.12, 1.14; [21], Theorem 1.6, (i), that any almost pro-$\mathcal{C}$-maximal quotient of G_K is strongly sn-internally indecomposable. In particular, by Lemma 2.3, we conclude that any almost pro-p-maximal quotient of G_K is sn-elastic.

Finally, we verify that any almost pro-$\mathcal{C}$-maximal quotient of G_K is sn-elastic. In light of Lemma 2.3, it suffices to prove that $G_K^{\mathcal{C}}$ is sn-elastic. Then it follows from Lemma 2.8, together with local class field theory, that $G_K^{\mathcal{C}}$ is sn-elastic. This completes the proof of Theorem 3.8. □

Lemma 3.9. *Let K be a Hilbertian field [cf. [3], §12.1]; $H \subseteq G_K$ a nontrivial subnormal closed subgroup; $U \subsetneq H$ a proper open subgroup of H. Write L for the separable extension of K associated to $U \subseteq G_K$. Then L is Hilbertian.*

Proof. There exist nonnegative integer n and distinct closed subgroups $H_0 = G, H_1, \ldots, H_{n-1}, H_n = H$ of G such that H_i is normal in H_{i-1} for each $i \in \{1, \ldots, n\}$. For each $i \in \{1, \ldots, n\}$, take a proper open subgroup $U_{i-1} \subsetneq H_{i-1}$ of H_{i-1} containing H_i, and write L_{i-1} for the separable extension of K associated to $U_{i-1} \subseteq G_K$. Moreover, write $L_n \stackrel{\mathrm{def}}{=} L$. Then, by applying [3], Theorem 13.9.1, (b), inductively, we observe that L_i is Hilbertian for each $i \in \{0, \ldots, n\}$. This completes the proof of Lemma 3.9. □

Theorem 3.10. *Let K be a Hilbertian field; $\mathcal{C}$ a nontrivial full-formation. Then any almost pro-$\mathcal{C}$-maximal quotient of G_K is strongly sn-internally indecomposable and very sn-elastic.*

Proof. We may assume that $p \in \Sigma_{\mathcal{C}}$. First, we claim the following:

> Claim 3.10.A: G_K^p is very sn-elastic.

Indeed, let $U \subseteq G_K^p$ be an open subgroup; $H \subseteq U$ a nontrivial subnormal closed subgroup of U. Take a proper open subgroup $V \subsetneq H$ of H. Then it follows from Lemma 3.9; [3], Corollary 16.3.6, that U, hence also H, is not topologically finitely generated. This completes the proof of Claim 3.10.A.

Next, we claim the following:

> Claim 3.10.B: Any almost pro-p-maximal quotient of G_K is strongly internally indecomposable.

Indeed, let Q be an almost pro-p-maximal quotient of G_K. In light of [3], Corollary 12.2.3, it suffices to prove that Q is internally indecomposable.

Write K_Q for the Galois extension of K associated to the kernel of the quotient map to Q [i.e., $\mathrm{Gal}(K_Q/K) = Q$]. Since K_Q has no nontrivial p-extension, K_Q is not Hilbertian. Thus, we observe that Q has no nontrivial finite normal subgroup [cf. [3], Theorem 13.9.1, (b)]. In particular, it follows from the proof of [14], Proposition 1.7, that we may assume that $Q = G_K^p$.

Now let $N \subseteq Q = G_K^p$ be a nontrivial normal closed subgroup. Then it follows from Claim 3.10.A; [15], Proposition 1.2, that N is slim. In particular, $Z(N) = Z_Q(N) \cap N$ is trivial. Thus, since K_Q is not Hilbertian, we conclude from [3], Theorem 13.8.3, that $Z_Q(N)$ is trivial, which implies that Q is internally indecomposable [cf. Proposition 1.7]. This completes the proof of Claim 3.10.B.

Next, we claim the following:

> Claim 3.10.C: Any almost pro-$\mathcal{C}$-maximal quotient of G_K is strongly sn-internally indecomposable.

Indeed, in light of Claim 3.10.B; Propositions 1.12, 1.14; [3], Corollary 12.2.3, it suffices to prove that G_K^p is sn-internally indecomposable. Let $H \subseteq G_K^p$ be a nontrivial subnormal closed subgroup of G_K^p. Take a proper open subgroup $U \subsetneq H$ of H. Observe that it follows from Claim 3.10.A; [15], Proposition 1.2, that H is slim. Moreover, it follows from Claim 3.10.B; Lemma 3.9, that U is strongly internally indecomposable. In particular, it follows from Proposition 1.12 that H is [strongly] internally indecomposable. Thus, we conclude that G_K^p is sn-internally indecomposable. This completes the proof of Claim 3.10.C.

To complete the proof of Theorem 3.10, it suffices to prove that any almost pro-$\mathcal{C}$-maximal quotient of G_K is very sn-elastic. This follows from Claims 3.10.A, 3.10.C; Lemmas 2.3, 2.4. This completes the proof of Theorem 3.10. $\square$

Next, we discuss the sn-internal indecomposability and the sn-elasticity of various groups appearing in anabelian geometry of hyperbolic curves.

Definition 3.11. Let K be a field; $\overline{K}$ an algebraic closure of K; X a smooth curve [i.e., a one-dimensional, smooth, separated, of finite type, and geometrically connected scheme] over K. Write $\overline{X}_{\overline{K}}$ for the smooth compactification of $X_{\overline{K}}$ over $\overline{K}$. Then we shall say that X is a *smooth curve of type* (g, r) over K if the genus of $\overline{X}_{\overline{K}}$ is g, and the cardinality of the underlying set of $\overline{X}_{\overline{K}} \setminus X_{\overline{K}}$ is r. If X is a smooth curve of type (g, r) over K, and $2g - 2 + r > 0$, then we shall say that X is a *hyperbolic curve* over K.

Definition 3.12 ([22], Definition 1.2)**.** Let $\mathcal{C}$ be a full-formation; Π a profinite group; Σ a nonempty set of prime numbers; l a prime number. Then we shall say that Π is a *pro-$\mathcal{C}$ surface group* (respectively, an *almost pro-$\mathcal{C}$ surface group*) if Π is isomorphic to the maximal pro-$\mathcal{C}$ quotient (respectively, an almost pro-$\mathcal{C}$-maximal quotient) of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic 0. If $\mathcal{C}$ is the family of all Σ-groups (respectively, all l-groups), then we shall also refer to a pro-$\mathcal{C}$ surface group as a *pro-Σ surface group* (respectively, a *pro-l surface group*).

Theorem 3.13. *Let $\mathcal{C}$ be a nontrivial full-formation; Π an almost pro-$\mathcal{C}$ surface group. Then Π is strongly sn-internally indecomposable and sn-elastic.*

Proof. Note that a pro-p surface group is a free pro-p group of rank ≥ 2 or a Demuškin group of rank ≥ 4. Thus, it follows from Propositions 1.12, 1.14; Theorems 3.3, 3.7; [22], Proposition 1.4, that Π is strongly sn-internally indecomposable. Moreover, it follows from Lemma 2.8; Theorems 3.3, 3.7; Proposition 3.6, (i), that Π is sn-elastic. This completes the proof of Theorem 3.13. $\square$

Corollary 3.14. *Let X be a hyperbolic curve over $\mathbb{C}_p$. Then $\pi_1^{\mathrm{temp}}(X)$ is sn-internally indecomposable.*

Proof. This follows from Proposition 1.10; Theorem 3.13; [1], Proposition 4.4.1; [1], §4.5. $\square$

Now we recall the definition of a configuration space group which plays a central role in combinatorial anabelian geometry [cf. [18], [19], [20], [6], [7], [8], [9], [10]].

Definition 3.15.

(i) Let K be a field; X a hyperbolic curve over K; n a positive integer. Write

$$X_n \stackrel{\text{def}}{=} X^{\times n} \setminus \Big(\bigcup_{1 \leq i < j \leq n} \Delta_{i,j} \Big),$$

where $X^{\times n}$ denotes the fiber product of n copies of X over K; $\Delta_{i,j}$ denotes the diagonal divisor of $X^{\times n}$ associated to the i-th and j-th components. We shall refer to X_n as the *n-th configuration space* associated to X.

(ii) Let $\mathcal{C}$ be a full-formation; Π a profinite group. Then we shall say that Π is a *pro-$\mathcal{C}$ configuration space group* if Π is isomorphic to the maximal pro-$\mathcal{C}$ quotient of the étale fundamental group of a configuration space associated to a hyperbolic curve over an algebraically closed field of characteristic 0.

Theorem 3.16. *Let $\mathcal{C}$ be a full-formation; Π a pro-$\mathcal{C}$ configuration space group. Suppose that either $\mathcal{C}$ is the family of all finite groups or $\Sigma_{\mathcal{C}}$ consists of a single element. Then Π is strongly sn-internally indecomposable.*

Proof. This follows from Proposition 1.16; Theorem 3.13; [2], Theorem 1; [2], Remark following the proof of Theorem 1. □

Remark 3.16.1. In the notation of Theorem 3.16, suppose that $\Pi \cong \pi_1(X_n)^{\mathcal{C}}$, where n is a positive integer and X is a hyperbolic curve over an algebraically closed field of characteristic 0. Then, since the kernel of the [outer] homomorphism $\Pi \to \pi_1(X)^{\mathcal{C}}$ determined by a projection $X_n \to X$ is topologically finitely generated [cf. [22], Remark 2.4.1], Π is not quasielastic whenever $n \geq 2$. In particular, the condition $n = 1$ is equivalent to the condition that Π is quasielastic, which is also equivalent to the condition that Π is sn-elastic [cf. Theorem 3.13].

In fact, it is known that n can precisely be reconstructed from Π [cf. [5], Theorem 2.5, (i); [28], Theorem A].

Corollary 3.17. *Let X be a hyperbolic curve over $\mathbb{C}$; n a positive integer. Then $\pi_1^{\mathrm{top}}(X_n)$ is sn-internally indecomposable.*

Proof. This follows from Proposition 1.10; Theorem 3.16; [22], Proposition 7.1, (ii). □

Lemma 3.18 (cf. [32], Lemma 1.10, (i); [12], Lemma 3.3)**.** *Let X be a smooth curve of type (g, r) over an algebraically closed field of characteristic p; l a prime number such that $l \neq p$. Suppose that $g \leq 1$ and $(g, r) \neq (0, 0), (1, 0)$ (respectively, $2g - 2 + r > 0$). Then there exists a normal open subgroup $N \subseteq \pi_1(X)$ such that $N \subseteq \pi_1(X)$ is of index p (respectively, of index a power of l), and that [the smooth compactification of] the domain curve of the covering associated to $N \subseteq \pi_1(X)$ has genus ≥ 2.*

Proof. The case where $g \leq 1$ and $(g, r) \neq (0, 0), (1, 0)$ follows immediately from the proof of [12], Lemma 3.3. The case where $2g - 2 + r > 0$ follows easily from the Hurwitz formula. □

Theorem 3.19. *Let Σ be a set of prime numbers such that $p \in \Sigma$; X a smooth curve of type (g, r) over an algebraically closed field of characteristic p. Suppose* *that $(g, r) \neq (0,0), (1,0)$. If $r = 0$ and $\Sigma = \{p\}$, then suppose that the p-rank $\sigma(X)$ of [the Jacobian variety of] X is not equal to 1. Then $\pi_1(X)^\Sigma$ is strongly sn-internally indecomposable and sn-elastic.*

Proof. If $\Sigma = \{p\}$, then it follows from [31], Theorem 4.9.4, that $\pi_1(X)^\Sigma$ is a free pro-p group of infinite rank (respectively, of rank $\sigma(X)$) if $r \neq 0$ (respectively, $r = 0$), which is strongly sn-internally indecomposable and sn-elastic [cf. Theorem 3.3]. Thus, to verify Theorem 3.19, we may assume that $\Sigma \supsetneq \{p\}$. First, we verify that $\pi_1(X)^\Sigma$ is strongly sn-internally indecomposable. It follows from the proof of [12], Theorem 3.6, that $\pi_1(X)^\Sigma$ is slim. In particular, in light of Proposition 1.12; Lemma 3.18, we may assume that $g \geq 2$.

Let $l \in \Sigma \setminus \{p\}$; Q an almost pro-l-maximal quotient of $\pi_1(X)^\Sigma$. Then it suffices to prove that Q is strongly sn-internally indecomposable [cf. Proposition 1.14]. Now there exists a finite Galois covering $Y \to X$ that determines an exact sequence of profinite groups

$$1 \to \pi_1(Y)^l \to Q \to \mathrm{Gal}(Y/X) \to 1.$$

Then it follows from [31], Theorem 4.9.1 that $\pi_1(Y)^l$ is a pro-l surface group, which is strongly sn-internally indecomposable [cf. Theorem 3.13]. Moreover, it follows from [12], Lemma 3.4, that the outer representation $\mathrm{Gal}(Y/X) \to \mathrm{Out}(\pi_1(Y)^l)$ associated to the above exact sequence is injective, which implies that Q is slim. Thus, we conclude from Proposition 1.12 that Q is strongly sn-internally indecomposable.

Next, we verify that $\pi_1(X)^\Sigma$ is sn-elastic [under the assumption $\Sigma \supsetneq \{p\}$]. Since $\pi_1(X)^\Sigma$ is strongly sn-internally indecomposable, in light of Lemmas 2.3, 3.18, we may assume that $g \geq 2$. Take l, Q, Y as above. Then it follows from Theorem 3.13 that $\pi_1(Y)^l$ is sn-elastic. Thus, since [we have already verified that] Q is strongly sn-internally indecomposable, it follows from Lemma 2.3 that Q is sn-elastic. Finally, it follows from Lemma 2.8; [31], Theorem 4.9.1, that $\pi_1(X)^\Sigma$ is sn-elastic. This completes the proof of Theorem 3.19. □

Remark 3.19.1. In the notation of Theorem 3.19, if $r = 0$ and $\sigma(X) = 1$, then $\pi_1(X)^p \cong \mathbb{Z}_p$, which is not [strongly sn-]internally indecomposable.

Theorem 3.20. *Let K be a field; X a smooth curve of type (g, r) over K; n a positive integer. Then the following hold:*

- (i) *Suppose that X is a hyperbolic curve over K, and that K is a number field or a p-adic local field. Then $\pi_1(X_n)$ is strongly sn-internally indecomposable.*
- (ii) *Suppose that X is a hyperbolic curve over K, and that K is a p-adic local field. Then $\pi_1^{\mathrm{temp}}(X)$ is sn-internally indecomposable.*
- (iii) *Suppose that $(g, r) \neq (0,0), (1,0)$, and that K is a finite field of characteristic p. Then $\pi_1(X)$ is strongly sn-internally indecomposable. If further suppose that $2g-2+r > 0$, then $\pi_1(X)/\ker(\pi_1(X_{\overline{K}}) \twoheadrightarrow \pi_1(X_{\overline{K}})^\Sigma)$, where Σ denotes the set of all prime numbers not equal to p, is strongly sn-internally indecomposable.*

Proof. Assertion (i) follows immediately from Proposition 1.16; Theorems 3.8, 3.10, 3.16; [6], Theorem C, (ii). Assertion (ii) follows from assertion (i); Proposition 1.10; [1], Proposition 4.4.1; [1], §4.5. Next, we verify assertion (iii). Let Σ be as in assertion (iii). Write $\Pi^{[\Sigma]} \overset{\mathrm{def}}{=} \pi_1(X)/\ker(\pi_1(X_{\overline{K}}) \twoheadrightarrow \pi_1(X_{\overline{K}})^\Sigma)$. It follows from Theorems 3.13, 3.19, that $\pi_1(X_{\overline{K}}), \pi_1(X_{\overline{K}})^\Sigma$ is strongly sn-internally indecomposable. Thus, since $G_K \cong \widehat{\mathbb{Z}}$ is abelian, in light of Proposition 1.17, it suffices to prove that the outer representations $\rho : G_K \to \mathrm{Out}(\pi_1(X_{\overline{K}}))$, $\rho^\Sigma : G_K \to \mathrm{Out}(\pi_1(X_{\overline{K}})^\Sigma)$ determined by the exact sequences

$$1 \to \pi_1(X_{\overline{K}}) \to \pi_1(X) \to G_K \to 1,\ 1 \to \pi_1(X_{\overline{K}})^\Sigma \to \Pi^{[\Sigma]} \to G_K \to 1$$

are injective. Since G_K is torsion-free, by applying Lemma 3.18; [12], Lemma 1.7, (i), we may assume that $g \geq 2$. Note that it suffices to prove that ρ^Σ is injective.

Write $\overline{X}$ for the smooth compactification of X over K. Then since

$$\mathrm{Hom}(H^2(\pi_1(\overline{X}_{\overline{K}})^\Sigma, \widehat{\mathbb{Z}}^\Sigma), \widehat{\mathbb{Z}}^\Sigma) \xrightarrow{\sim} \widehat{\mathbb{Z}}^\Sigma(1)$$

as G_K-modules, where "(1)" denotes the Tate twist, we conclude that ρ^Σ is injective. This completes the proof of assertion (iii), hence also of Theorem 3.20. □

Remark 3.20.1. In the notation of Theorem 3.20, (i) (respectively, (iii)), the normal closed subgroup $\pi_1((X_n)_{\overline{K}})$ (respectively, $\pi_1(X_{\overline{K}})^\Sigma)$) of $\pi_1(X_n)$ (respectively, $\pi_1(X)/\ker(\pi_1(X_{\overline{K}}) \twoheadrightarrow \pi_1(X_{\overline{K}})^\Sigma))$ is topologically finitely generated [cf. [22], Proposition 2.2, (ii); [31], Theorem 4.9.1] and of infinite index. In particular, $\pi_1(X_n)$ (respectively, $\pi_1(X)/\ker(\pi_1(X_{\overline{K}}) \twoheadrightarrow \pi_1(X_{\overline{K}})^\Sigma))$ is not quasielastic.

Remark 3.20.2. In the present remark, we shall use the language of combinatorial anabelian geometry [cf. [18], [19], [20], [6], [7], [8], [9], [10]]. Recall that the notion of an outer representation of NN-type plays a central role. Let Σ be a nonempty set of prime numbers; $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type such that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$. Write $\Pi_\mathcal{G}$ for the fundamental group of $\mathcal{G}$. Note that $\Pi_\mathcal{G}$ may be identified with a pro-Σ surface group. Let

$$\rho : I \to \mathrm{Out}(\Pi_\mathcal{G})$$

be an outer representation of pro-Σ PSC-type. Then ρ determines an exact sequence of profinite groups

$$1 \to \Pi_\mathcal{G} \to \Pi_I \stackrel{\mathrm{def}}{=} \Pi_\mathcal{G} \stackrel{\mathrm{out}}{\rtimes} I \to I \to 1.$$

Suppose that ρ is of NN-type. Then it holds that $I \cong \widehat{\mathbb{Z}}^\Sigma$, and ρ is injective [cf. our assumption that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$]. Thus, it follows immediately from Theorem 3.13, together with Proposition 1.17, that Π_I is strongly sn-internally indecomposable.

Acknowledgements

The authors would like to thank the referee for valuable comments. The first author was supported by JSPS KAKENHI Grant Number JP20K14285. The second author was supported by JSPS KAKENHI Grant Numbers JP20J00323, JP22K13892. This research was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University, as well as by the International Research Center for Next Generation Geometry [a research center affiliated with the Research Institute for Mathematical Sciences].

References

[1] Y. André, On a geometric description of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a p-adic avatar of $\widehat{\mathrm{GT}}$, *Duke Math. J.* **119** (2003), no. 1, 1–39.

[2] M. Asada, The faithfulness of the monodromy representations associated with certain families of algebraic curves, *J. Pure Appl. Algebra* **159** (2001), no. 2–3, 123–147.

[3] M. Fried and M. Jarden, *Field arithmetic*, Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge. A Series of Modern Surveys in Mathematics **11**, Springer-Verlag, Berlin, 2008.

[4] Y. Hoshi, The Grothendieck conjecture for hyperbolic polycurves of lower dimension, *J. Math. Sci. Univ. Tokyo* **21** (2014), no. 2, 153–219.

[5] Y. Hoshi, A. Minamide, and S. Mochizuki, Group-theoreticity of numerical invariants and distinguished subgroups of configuration space groups, *Kodai Math. J.* **45** (2022), no. 3, 295–348.

[6] Y. Hoshi and S. Mochizuki, On the combinatorial anabelian geometry of nodally nondegenerate outer representations, *Hiroshima Math. J.* **41** (2011), no. 3, 275–342.

[7] Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: inertia groups and profinite Dehn twists, *Galois-Teichmüller theory and arithmetic geometry*, *Adv. Stud. Pure Math.* **63**, *Math. Soc. Japan*, 2012, 659–811.

[8] Y. Hoshi and S. Mochizuki, *Topics surrounding the combinatorial anabelian geometry of hyperbolic curves II: Tripods and combinatorial cuspidalization*, Lecture Notes in Mathematics, **2299**, Springer, Singapore, 2022.

[9] Y. Hoshi and S. Mochizuki, *Topics surrounding the combinatorial anabelian geometry of hyperbolic curves III: Tripods and Tempered fundamental groups*, to appear in *Kyoto J. Math.*

[10] Y. Hoshi and S. Mochizuki, *Topics surrounding the combinatorial anabelian geometry of hyperbolic curves IV: Discreteness and Sections*, to appear in *Nagoya Math. J.*

[11] A. Ivanov, On some anabelian properties of arithmetic curves, *Manuscripta Math.* **144** (2014), no. 3–4, 545–564.

[12] A. Minamide, Indecomposability of various profinite groups arising from hyperbolic curves, *Math. J. Okayama Univ.* **60** (2018), 175–208.

[13] A. Minamide and S. Tsujimura, Anabelian group-theoretic properties of the absolute Galois groups of discrete valuation fields, *J. Number Theory* **239** (2022), 298–334.

[14] A. Minamide and S. Tsujimura, Internal indecomposability of profinite groups, *Adv. Math.* **409** (2022), Paper No. 108689.

[15] A. Minamide and S. Tsujimura, *Anabelian group-theoretic properties of the pro-p absolute Galois groups of Henselian discrete valuation fields*, RIMS Preprint **1952** (August 2021).

[16] A. Minamide and S. Tsujimura, *Anabelian geometry for Henselian discrete valuation fields with quasi-finite residues*, RIMS Preprint **1973** (June 2023).

[17] S. Mochizuki, The local pro-p anabelian geometry of curves, *Invent. Math.* **138** (1999), no. 2, 319–423.

[18] S. Mochizuki, Semi-graphs of anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), no. 1, 221–322.

[19] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), no. 3, 455–479.

[20] S. Mochizuki, On the combinatorial cuspidalization of hyperbolic curves, *Osaka J. Math.* **47** (2010), no. 3, 651–715.

[21] S. Mochizuki, Topics in absolute anabelian geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), no. 2, 139–242.

[22] S. Mochizuki and A. Tamagawa, The algebraic and anabelian geometry of configuration spaces, *Hokkaido Math. J.* **37** (2008), no. 1, 75–131.

[23] S. Mochizuki and S. Tsujimura, *Resolution of nonsingularities, point-theoreticity, and metric-admissibility for p-adic hyperbolic curves*, RIMS Preprint **1974** (June 2023).

[24] H. Nakamura, Galois rigidity of the étale fundamental groups of punctured projective lines, *J. Reine Angew. Math.* **411** (1990), 205–216.

[25] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, Second edition, Grundlehren der mathematischen Wissenschaften. A Series of Comprehensive Studies in Mathematics **323**, Springer-Verlag, Berlin, 2008.

[26] L. Ribes and P. Zalesskii, *Profinite Groups*, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics **40**, Springer-Verlag, Berlin, 2010.

[27] D. J. S. Robinson, *A Course in the Theory of Groups*, Second edition, Graduate Texts in Mathematics, **80**, Springer-Verlag, New York, 1996.

[28] K. Sawada, Cohomology of the geometric fundamental group of hyperbolic polycurves, *J. Algebra* **508** (2018), 364–389.

[29] K. Sawada, Pro-p Grothendieck conjecture for hyperbolic polycurves, *Publ. Res. Inst. Math. Sci.* **54** (2018), no. 4, 781–853.

[30] K. Sawada, *Reconstruction of invariants of configuration spaces of hyperbolic curves from associated Lie algebras*, RIMS Preprint **1896** (November 2018).

[31] T. Szamuely, *Galois Groups and Fundamental Groups*, Cambridge Stud. Adv. Math. **117**, Cambridge Univ. Press, Cambridge, 2009.

[32] A. Tamagawa, The Grothendieck conjecture for affine curves, *Compositio Math.* **109** (1997), no. 2, 135–194.

(Arata Minamide) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: `minamide@kurims.kyoto-u.ac.jp`

(Koichiro Sawada) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: `sawada@kurims.kyoto-u.ac.jp`

(Shota Tsujimura) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: `stsuji@kurims.kyoto-u.ac.jp`